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Inviscid radiating flow over a blunt body

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The effect of thermal radiation is investigated for the axisymmetric flow over the blunt body associated with a given paraboloidal shock wave. Radiative transfer is treated by means of the differential approximation, which applies to multidimensional flow and is valid throughout the entire range of temperature and optical thickness. The gas is assumed to be perfect and optically grey, and molecular-transport processes are neglected. A semi-analytical solution for the flow and radiation fields is obtained by the method of series truncation.

Results are presented, in the strong-shock approximation, for various values of the appropriate dimensionless variables. In general, radiation is found to have a significant influence on temperature and density, moderate effect on velocity, and little effect on pressure. The stand-off distance between the shock wave and the body is found to decrease significantly with increasing radiation; the body shape is less affected. The anomalous behaviour of the gas temperature on the body streamline as obtained by earlier investigators in the optically thin case does not appear in the present work. The results thus show correct physical behaviour throughout the flow field for all values of optical thickness. The detailed flow quantities exhibit a number of features of multidimensional radiating flow. They also provide a check on the special assumptions made in other, more approximate treatments. Similarities between radiating flow and non-equilibrium reactive flow over blunt bodies are apparent.

1. Introduction

A considerable amount of work has been done recently on the problem of the radiating flow over a blunt body. The problem has important application to blunt-nosed re-entry vehicles when the heat transfer due to radiation is no longer negligible. Since an exact treatment taking full account of the multidimensional character of the radiation field is extremely difficult, simplifying assumptions and approximations are usually made.

One common assumption is to regard the radiative field as one-dimensional while still accounting for the reabsorption of radiation by the gas. For example, Yoshikawa & Chapman (1962) represented the shock layer in front of a blunt body by studying the inviscid flow through a normal shock wave and into a porous planar wall. Both the flow field and radiation field in this case are truly one-dimensional. An analytical method of successive approximation is then used

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to evaluate the troublesome reabsorption integral in the energy equation. Later, Howe & Viegas (1964) considered the flow near the stagnation region of a solid convex body. In this work only the radiation field is treated as one-dimensional. The flow in the entire shock layer is taken to be viscous and heat-conducting, and the usual assumption of local similarity is employed. The reabsorption integral is evaluated numerically. At about the same time, Goulard (1964) analysed the inviscid stagnation flow, also on the basis of a one-dimensional treatment of the radiative field. He observed that when the appropriate radiation-convection parameter is small, the problem can be treated as a perturbation on the non-radiating flow. Explicit analytical solutions can be obtained if, in addition, the thin-gas approximation is made. A qualitative description of the shock layer for a thick gas is also given. Goulard shows that in the latter case the temperature distribution across the layer has a 'boundary-layer-type' behaviour, both adjacent to the shock wave and to the wall; between the two layers there is a region of isothermal flow. Most recently, Olstad (1965) has given a comprehensive study of essentially the same problem as Goulard, using four different approaches for four different situations in the gas. Of particular interest is the case in which the effects of radiation can be treated as a small perturbation on the non-radiating flow. Olstad finds that in this case a regular perturbation procedure is invalid near the wall. He is able to obtain a uniformly valid solution by means of the Lighthill technique.

An essentially different approach is to assume at the outset that the gas is optically thin. If the body and free stream do not radiate too intensively, reabsorption by the gas can then be neglected and the effect of radiation represented by a distributed system of heat sinks. The multidimensional character of the radiative field can then be retained without prohibitive difficulty. Using this approach, Wilson & Hoshizaki (1965) considered the direct problem of the inviscid flow over blunt bodies using the integral method of Maslen & Moeckel (1957). In this approach, the temperature and tangential-velocity profiles are expressed as power series in the Dorodnitsyn variable. The results show that although the velocity distribution and shock shape are not sensitive to the profile selected, the enthalpy distribution and stand-off distance are. Wang (1965) similarly studied the inviscid radiating flow over a sphere, but on the basis of the mathematical methods of Freeman (1956) and Chernyi (1961), which assume a geometrically thin shock layer. In both of these studies the temperature of the gas on the body streamline is found to be identically equal to the assumedly zero temperature of the body. This anomalous behaviour—anomalous because there is no real reason why it should be so in the absence of molecular-transport processes—is a consequence of the neglect of reabsorption in the thin-gas approximation. The thin-gas solution is thus invalid in the strongly cooled region near the body.

To obtain a uniformly valid solution, it is necessary to retain the reabsorption effects, at least adjacent to the body. Returning to the earlier one-dimensional approach and using a model similar to that of Goulard, Thomas (1965) showed that the incorrect behaviour near the wall can be removed by expanding the temperature in the reabsorption integral in a Taylor's series and retaining the first two terms. For small and modest values of the radiation-convection parameter, Thomas's results show realistic behaviour throughout the flow field, and the gas temperature at the wall has a well-defined non-zero value. For large values of the parameter, higher-order terms in the expansion must be retained.

All the foregoing work is restricted either to the stagnation region with the assumption of a one-dimensional radiative field or to the thin-gas approximation with consequent incorrect results near the body streamline. The problem remains of treating multidimensional and reabsorption effects at the same time. Recent developments suggest that this can perhaps best be done on the basis of the so-called 'differential approximation' of radiative transfer. In this approach, which was introduced in astrophysics and has since been highly developed in neutron-transport theory, the exact integro-differential equations of radiative transfer in three dimensions are replaced by an approximate set of purely differential equations valid for the full range of temperature and optical thickness (see, e.g. Traugott 1963; Cheng 1964, 1965, 1966). This method was used to solve the two-dimensional problem of the linearized flow over a wavy wall by Cheng (1966). The present paper seeks to apply the same approach to the nonlinear radiating flow over a blunt body.

The mathematical method that will be used for the solution is the method of series truncation developed by Swigart (1963), Kao (1964), Van Dyke (1965), and Conti (1966). In this method the dependent variables in the flow and radiation equations are expanded in power series in a longitudinal curvilinear co-ordinate away from the stagnation streamline. A closed set of ordinary differential equations is obtained by truncating the power series, and these equations are solved by standard numerical techniques. The method is here applied to the inverse problem of the flow behind a given paraboloidal shock wave. Detailed results are presented, as obtained from the second-order truncation, for a number of values of the dimensionless parameters governing the radiative flow.

Coincident with the completion of this work, a paper was presented by Wang (1966), who also uses the differential approximation to treat radiating flow over symmetric bodies. Again following the method of Freeman (1956) for a geometrically thin shock layer, Wang expands the dependent variables as power series in the density ratio across the shock. For the zeroth approximation, which is as far as the solutions are carried, the multidimensional radiative equations reduce to one-dimensional form, and the velocity and pressure are the same as for a non-radiating gas. The temperature on the body streamline is found to approach a well-defined non-zero value on a wedge or cone but is zero on a sphere. Wang attributes this behaviour to the inaccurate velocity profile given for the sphere by the zeroth approximation. Wang's mathematical methods and approximations are thus entirely different from those employed here.

2. Governing equations and boundary conditions

To simplify the problem and concentrate our attention on the effects of radiation, we make the following assumptions regarding the gas model:

(*a*) Non-equilibrium effects from all processes other than absorption and emission of radiation are negligible. We ignore in particular all non-equilibrium molecular phenomena.

(b) The direct contributions of radiation to pressure and internal energy are negligible.

(c) The gas is thermally and calorically perfect.

(d) Scattering of radiation is negligible and the gas can be treated as grey (i.e. absorption coefficient independent of wave length). The absorption coefficient, however, is taken to depend on the local thermodynamic state.

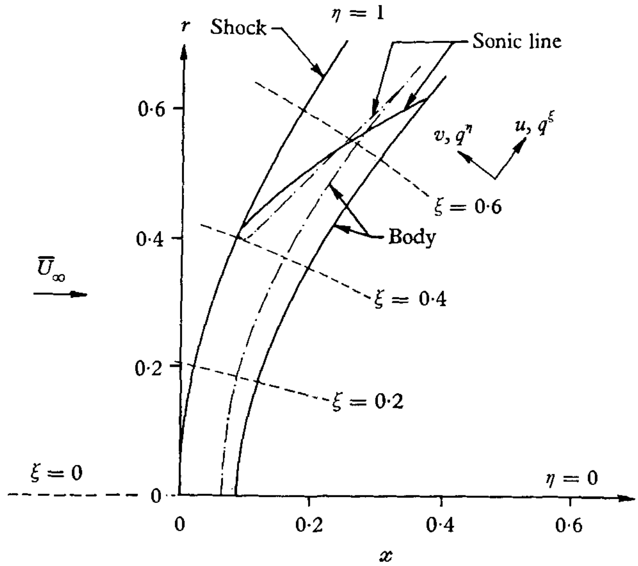

FIGURE 1. Effect of radiation on stand-off distance, sonic line, and body shape; ——, non-radiating ($Bu = 0$); —·—, radiating ($\Gamma = 0{\cdot}4$, $Bu = 2{\cdot}7$).

We shall be concerned specifically with the inverse problem of the axisymmetric flow field over the blunt body associated with a paraboloidal shock wave. The cold gas ahead of the shock wave is assumed to be neither absorbing nor emitting. All radiation that passes out through the shock wave thus escapes to infinity, and the flow ahead of the wave can therefore be taken as uniform. The shock wave, being of zero thickness in the absence of molecular-transport effects, is itself also transparent to radiation. The surface of the body is taken to be black and at constant temperature.

The formulation of the problem will follow closely that of Van Dyke (1965) for a non-radiating gas. As there, the geometry of the problem suggests a paraboloidal system of co-ordinates (Van Dyke 1958). The dimensionless paraboloidal co-ordinates ξ and η are related in particular to the dimensional cylindrical co-ordinates $\bar{x}$ and $\bar{r}$ by $x \equiv \bar{x}/\bar{R}_s = \frac{1}{2}(\xi^2 - \eta^2 + 1)$ and $r \equiv \bar{r}/\bar{R}_s = \xi\eta$, where $\bar{R}_s$ is the nose radius of the shock wave, which lies in the surface $\eta = 1$ (see figure 1). (Throughout the paper dimensional quantities are denoted with bars and the corresponding dimensionless quantities without. Quantities pertaining to the point immediately behind the shock on the stagnation streamline are denoted by subscript s.) The surfaces ξ = constant and η = constant are orthogonal to each other and are confocal paraboloids of revolution with focus at $x = \frac{1}{2}$, $r = 0$.

In writing the equations of motion it is convenient to refer the velocity components to the free-stream speed $\overline{U}_\infty$, density to its free-stream value $\overline{\rho}_\infty$, pressure to $\overline{\rho}_\infty \overline{U}^2_\infty$, temperature to the temperature $\overline{T}_s$ behind the shock on the stagnation streamline ($\xi = 0$, $\eta = 1$), and all radiative quantities to $\sigma \overline{T}^4_s$, where σ is the Stefan–Boltzmann constant. If u and v are the dimensionless velocity components in the ξ and η directions (figure 1) and ρ is the dimensionless density, the continuity equation in the paraboloidal co-ordinates is

$$[\xi\eta(\xi^2+\eta^2)^{\frac{1}{2}}\rho u]_\xi + [\xi\eta(\xi^2+\eta^2)^{\frac{1}{2}}\rho v]_\eta = 0, \tag{1}$$

where the subscript denotes partial differentiation with respect to the indicated variable. Under assumptions (a) and (b), the momentum equations in the ξ and η directions are

$$uu_\xi + v\left(u_\eta - \frac{\xi v - \eta u}{\xi^2+\eta^2}\right) + \frac{p_\xi}{\rho} = 0, \tag{2}$$

$$u\left(v_\xi + \frac{\xi v - \eta u}{\xi^2+\eta^2}\right) + vv_\eta + \frac{p_\eta}{\rho} = 0, \tag{3}$$

where p is the dimensionless pressure. With assumptions (a), (b) and (c), the energy and state equations can be written

$$\left(\frac{\gamma}{\gamma-1}\right)\frac{p}{T}(uT_\xi + vT_\eta) - (up_\xi + vp_\eta) + \frac{\Gamma}{\xi\eta(\xi^2+\eta^2)^{\frac{1}{2}}}\{[\xi\eta(\xi^2+\eta^2)^{\frac{1}{2}} q^\xi]_\xi + [\xi\eta(\xi^2+\eta^2)^{\frac{1}{2}} q^\eta]_\eta\} = 0, \tag{4}$$

and

$$p = \rho T/\gamma M_r^2, \tag{5}$$

where T is the dimensionless temperature, q^ξ and q^η the components of the dimensionless radiative heat flux in the ξ and η directions (figure 1), γ the ratio of specific heats, and M_r a 'mixed' Mach number defined by $M_r \equiv \overline{U}_\infty/(\gamma R\overline{T}_s)^{\frac{1}{2}}$, where R is the ordinary gas constant. The dimensionless parameter $\Gamma \equiv \sigma\overline{T}^4_s/\overline{\rho}_\infty \overline{U}^3_\infty$ that appears in (4) is the ratio of the black-body heat flux at the temperature $\overline{T}_s$ to twice the flux of kinetic energy in the free stream. It is a measure of the relative importance of the radiative and convective processes in the flow of energy.

As mentioned in the introduction, the radiative heat flux in (4) is taken to be governed by the equations of the differential approximation of radiative transfer (Traugott 1963; Cheng 1964, 1965, 1966; for a general expository discussion see also Vincenti & Kruger 1965). In this approximation, the exact integro-differential equations governing the heat flux are replaced by a set of moments (with respect to direction of propagation of the radiation) of the exact differential equation that governs the radiative intensity. The set is rendered finite with the aid of the Milne–Eddington approximation relating certain of the quantities that appear in the equations. The same set of equations can also be obtained as the first approximation in a method that expands the radiative intensity in a series in terms of spherical harmonics of the direction of propagation. Either way, an approximate set of purely differential equations is obtained. Implicit in this approach are assumptions (a) and (d) above, assumption (a) being described in radiative theory as the 'assumption of local thermodynamic equilibrium'. To write the resulting equations we assume that the grey volumetric absorption coefficient $\bar{\alpha}$ (with dimensions of reciprocal length) can be written in terms of pressure and temperature as $\bar{\alpha} = C\bar{p}^a\bar{T}^b = C(\bar{\rho}_\infty \bar{U}_\infty^2)^a \bar{T}_s^b p^a T^b$, where C, a and b are constants evaluated from experimental data. The equations of radiative transfer in the differential approximation, when written in the present coordinate system, are then (cf. Vincenti & Kruger 1965, p. 492)

$$[\xi\eta(\xi^2+\eta^2)^{\frac{1}{2}} q^\xi]_\xi + [\xi\eta(\xi^2+\eta^2)^{\frac{1}{2}} q^\eta]_\eta = -\beta p^a T^b \xi\eta(\xi^2+\eta^2)(I_0 - 4T^4), \tag{6}$$

$$I_{0_\xi} = -3\beta p^a T^b (\xi^2+\eta^2)^{\frac{1}{2}} q^\xi, \tag{7}$$

$$I_{0_\eta} = -3\beta p^a T^b (\xi^2+\eta^2)^{\frac{1}{2}} q^\eta, \tag{8}$$

where I_0 is the radiative intensity, integrated over all frequencies and over all directions passing through a given point (and equal to the speed of light times the radiative energy density). The dimensionless parameter $\beta \equiv \bar{R}_s C(\bar{\rho}_\infty \bar{U}_\infty^2)^a \bar{T}_s^b$ is a measure of the size of the flow field relative to a characteristic radiative mean free path in the shock layer. Equations (1)–(8) constitute eight equations for the eight unknowns u, v, p, T, ρ, I_0, q^ξ and q^η. The first five unknowns describe the flow field and the last three the radiative field, the two fields being coupled through the heat-flux terms in the energy equation (4) and the temperature and pressure in the radiative equations (6)–(8). When either $\beta = 0$ or $\Gamma = 0$ the heat-flux terms disappear in (4), and the flow and radiative fields become decoupled. The system of equations (1)–(5) can then be reduced to the equations given by Van Dyke (1965) for the non-radiating gas.

To simplify the boundary conditions at the shock wave, the temperature and pressure in the flow ahead of the shock are assumed to be negligible (strong-shock approximation). For our assumedly uniform flow ahead of the wave, the conditions on the flow quantities immediately behind the shock ($\eta = 1$) can then be written

$$u(\xi,1) = \xi/(1+\xi^2)^{\frac{1}{2}}, \quad v(\xi,1) = -(\gamma-1)/[(\gamma+1)(1+\xi^2)^{\frac{1}{2}}], \tag{9a,b}$$

$$p(\xi,1) = 2/[(\gamma+1)(1+\xi^2)], \quad T(\xi,1) = 1/(1+\xi^2), \quad \rho(\xi,1) = (\gamma+1)/(\gamma-1). \tag{9c,d,e}$$

Because of the nature of the radiative field, the radiative quantities immediately behind the shock are not known *a priori*, even when the shock shape is prescribed. Our assumption that the flow ahead of the shock does not emit, however, leads, within the differential approximation, to the following relation between the radiative quantities immediately behind the shock:

$$I_0(\xi,1) - 2q^\eta(\xi,1) = 0. \tag{10}$$

This relation is found by considering the one-sided radiative flux coming from the free-stream and passing through the shock wave, and requiring that the value of this heat flux calculated within the differential approximation be equal to its exact, assumed value of zero (cf. Cheng 1965, 1966; Vincenti & Kruger 1965, p. 495).

For an inverse problem, the shock shape is prescribed and the body shape is to be found. This is done by satisfying the condition that the normal component of velocity must vanish at the body surface, which can be written

$$v(\xi, \eta_w) - u(\xi, \eta_w)\,(d\eta_w/d\xi) = 0, \tag{11}$$

where $\eta_w = \eta_w(\xi)$ specifies the shape of the body, which is to be found. The radiative condition at the surface of the body is

$$\tfrac{1}{4}\{I_0(\xi, \eta_w) + 2[1 + (d\eta_w/d\xi)^2]^{-\frac{1}{2}}[q^\eta(\xi, \eta_w) - q^\xi(\xi, \eta_w)\,(d\eta_w/d\xi)]\} = T_w^4, \tag{12}$$

where T_w is the assumedly constant wall temperature. This is obtained by considering the one-sided radiative flux coming into the shock layer from the body and requiring that this flux calculated within the differential approximation be equal to the exact black-body value at the dimensionless wall temperature T_w.

The boundary conditions (9)–(12) are sufficient to determine the solution. There might at first be some doubt about this, since, if we consider the radiative field alone, (10) and (12) appear to constitute only two boundary conditions on the three unknowns I_0, q^ξ and q^η. Actually, the differential equation (7) in effect supplies a second condition at the shock wave when specialized to $\eta = 1$. Thus, if we imagine the flow field known and a solution for the radiative quantities proceeding by trial and error from guesses at the shock wave, only one such quantity can be guessed independently. For example, if we guess $q^\eta(\xi, 1)$, then $I_0(\xi, 1)$ follows from (10) and $q^\xi(\xi, 1)$ from (7) when that equation is specialized to $\eta = 1$. Satisfaction of just the one condition (12) at the body is therefore sufficient to pick out the correct variation of $q^\eta(\xi, 1)$ and thus determine the solution.

In the present formulation of radiating blunt-body flow, even the inverse problem is a two-sided boundary-value problem. This is in contrast to the situation in non-radiating flow or in radiating flow in the thin-gas approximation, where the inverse approach leads to a purely initial-value problem.

3. Method of series truncation

The method of series truncation was first applied by Swigart (1963) to treat the inverse problem of the inviscid two- and three-dimensional asymmetric non-radiating flow over blunt bodies. In Swigart's approach, the stream function ψ and density ρ, which are the primary dependent variables, are first expanded in power series in the longitudinal curvilinear co-ordinate ξ. Substitution of these series into the governing partial differential equations and collection of like powers of ξ then lead to a set of ordinary differential equations with the normal co-ordinate η as the independent variable. A closed set of equations is finally obtained by truncating the expansion series to appropriate order, and these equations are solved numerically. Results obtained on the stagnation streamline with a fourth-order truncation were shown to be accurate to at least four significant figures as compared with a direct numerical solution of the exact problem. Away from the stagnation streamline, the accuracy is not as good. For example, surface pressure at the sonic point is unreliable in the second significant figure.

Improved versions of the truncation scheme have been applied by Kao (1964) and Conti (1966) to viscous flow and to non-equilibrium reacting flow respectively. Conti (1966) found that if pressure rather than density is expanded in the power series, the accuracy of each truncation is greatly improved. In a recent paper, Van Dyke (1965) has incorporated Conti's idea of using pressure as a primary variable together with two modifications of Swigart's scheme, namely, (1) the use of $\xi^2/(1+\xi^2)$ instead of ξ as the expansion variable, and (2) the interchange of the roles of η and ψ. The second-order truncation in Van Dyke's approach yields five-figure accuracy on the stagnation streamline, and the fourth-order truncation gives four-figure accuracy throughout the subsonic region behind a paraboloidal shock. In view of the success of Van Dyke's scheme, we shall follow his procedures closely and extend them to the radiating gas.

As the first step, we introduce a stream function such that

$$\psi_\eta = \xi\eta(\xi^2+\eta^2)^{\frac{1}{2}}\rho u, \tag{13a}$$

and

$$\psi_\xi = -\xi\eta(\xi^2+\eta^2)^{\frac{1}{2}}\rho v, \tag{13b}$$

which satisfy the continuity equation (1) identically. From previous work on radiating shock layers, it is known that the effect of radiation on pressure is small. Thus the pressure, which in the strong-shock approximation has a Newtonian variation just behind the shock, still retains a similar variation along the body even in the presence of radiation. The same observation with regard to non-equilibrium reactive flow led Conti (1966) to suggest that pressure is to be preferred over density as the expansion variable. Clearly, Conti's argument is also applicable here. We therefore choose to eliminate ρ from the momentum equations (2) and (3) with the aid of the state equation (5). The resulting equations after substitution for u and v from equations (13) are

$$\psi_\eta\psi_{\xi\eta}-\psi_\xi\psi_{\eta\eta}+\psi_\eta\left[\psi_\xi\left(\frac{p_\eta}{p}-\frac{T_\eta}{T}+\frac{1}{\eta}\right)-\psi_\eta\left(\frac{p_\xi}{p}-\frac{T_\xi}{T}+\frac{1}{\xi}\right)\right]-\frac{\xi(\psi_\xi^2+\psi_\eta^2)}{(\xi^2+\eta^2)} + \frac{\gamma M_r^2 p p_\xi (\xi\eta)^2(\xi^2+\eta^2)}{T} = 0, \tag{14}$$

$$\psi_\xi\psi_{\xi\eta}-\psi_\eta\psi_{\xi\xi}+\psi_\xi\left[\psi_\eta\left(\frac{p_\xi}{p}-\frac{T_\xi}{T}+\frac{1}{\xi}\right)-\psi_\xi\left(\frac{p_\eta}{p}-\frac{T_\eta}{T}+\frac{1}{\eta}\right)\right]-\frac{\eta(\psi_\xi^2+\psi_\eta^2)}{(\xi^2+\eta^2)} + \frac{\gamma M_r^2 p p_\eta (\xi\eta)^2(\xi^2+\eta^2)}{T} = 0. \tag{15}$$

The energy equation (4) can be simplified by making use of the first radiation equation (6). The result in terms of ψ, p and T is

$$[\gamma/(\gamma-1)][\psi_\eta T_\xi-\psi_\xi T_\eta]-(T/p)[\psi_\eta p_\xi-\psi_\xi p_\eta] -\beta\Gamma\gamma M_r^2 p^a T^b \xi\eta(\xi^2+\eta^2)(I_0-4T^4) = 0. \tag{16}$$

Here, since the radiating flow is not isentropic, it is not possible as in Van Dyke's work to reduce the system to only two equations in ψ and p. Instead, the temperature T, as well as the radiative quantities, must be retained as primary variables.

The boundary conditions $(9a, b)$ can be replaced by two conditions on ψ. Integration of $(13b)$ along the shock ($\eta = 1$) following substitution from $(9b)$ and $(9e)$ gives, in particular,

$$\psi(\xi, 1) = \tfrac{1}{2}\xi^2. \tag{17a}$$

Substitution of (9a) and (9e) into (13a) with $\eta = 1$ leads to

$$\psi_\eta(\xi, 1) = \frac{(\gamma+1)}{(\gamma-1)}\xi^2. \tag{17b}$$

The counterpart of the boundary condition (11) on the body is

$$\psi(\xi, \eta_w) = 0. \tag{18}$$

Since velocities appear neither in the approximate radiation-transport equations (6) to (8) nor in the radiative boundary conditions (10) and (12), these equations are unaffected by the introduction of the stream function.

We now interchange the role of η and ψ by means of the Von Mises transformation, so that ψ becomes an independent variable and $\eta = \eta(\xi, \psi)$ a dependent variable. Equations (14) to (16) and (6) to (8) after such a transformation become

$$\eta_{\xi\psi} + \eta_\psi\left[\frac{p_\xi}{p} - \frac{T_\xi}{T} + \left(\frac{1}{\xi} + \frac{\eta_\xi}{\eta}\right)\right] + \xi\eta_\psi\frac{(1+\eta_\xi^2)}{(\xi^2+\eta^2)} + (\xi\eta)^2(\xi^2+\eta^2)\eta_\psi^2\frac{\gamma M_r^2}{T}p(\eta_\xi p_\psi - \eta_\psi p_\xi) = 0, \tag{19}$$

$$\eta_\psi\eta_{\xi\xi} - \eta_\xi\eta_{\xi\psi} - \eta_\psi\eta_\xi\left[\frac{p_\xi}{p} - \frac{T_\xi}{T} + \left(\frac{1}{\xi} + \frac{\eta_\xi}{\eta}\right)\right] - \frac{\eta\eta_\psi(1+\eta_\xi^2)}{(\xi^2+\eta^2)} + (\xi\eta)^2(\xi^2+\eta^2)\eta_\psi^2\frac{\gamma M_r^2}{T}pp_\psi = 0, \tag{20}$$

$$\frac{\gamma}{\gamma-1}T_\xi - \frac{Tp_\xi}{p} - \beta\Gamma\gamma M_r^2 p^a T^b \xi\eta(\xi^2+\eta^2)(I_0 - 4T^4)\eta_\psi = 0, \tag{21}$$

$$\eta\xi(\xi^2+\eta^2)(\eta_\psi q_\xi^\xi - \eta_\xi q_\psi^\xi + q_\psi^\eta) + (\xi^2+\eta^2)(\eta q^\xi + \xi q^\eta)\eta_\psi + \xi\eta(\xi q^\xi + \eta q^\eta)\eta_\psi + \beta p^a T^b(I_0 - 4T^4)(\xi\eta)(\xi^2+\eta^2)^{\frac{3}{2}}\eta_\psi = 0, \tag{22}$$

$$\eta_\psi I_{0\xi} - \eta_\xi I_{0\psi} + 3\beta p^a T^b(\xi^2+\eta^2)^{\frac{1}{2}} q^\xi \eta_\psi = 0 \tag{23}$$

and

$$I_{0\psi} + 3\beta p^a T^b(\xi^2+\eta^2)^{\frac{1}{2}} q^\eta \eta_\psi = 0. \tag{24}$$

Following Van Dyke (1965), we now normalize the stream function with respect to its value at the shock for the same value of ξ. Using (17a), we thus set

$$\psi \equiv \tfrac{1}{2}\xi^2\omega, \tag{25a}$$

where ω is the normalized stream function. In the same manner we normalize p and T relative to their values (9c, d) immediately behind the shock. We thus take

$$p \equiv 2(\gamma+1)^{-1}(1+\xi^2)^{-1}P(z, \omega), \tag{25b}$$

and

$$T \equiv (1+\xi^2)^{-1}\Theta(z, \omega), \tag{25c}$$

where P and Θ are the normalized pressure and temperature and $z \equiv \xi^2/(1+\xi^2)$ is a new independent variable as suggested by Van Dyke.

The radiative quantities immediately behind the shock are not known *a priori*. We do know, however, that q^ξ vanishes at the stagnation streamline ($\xi = 0$) and must be antisymmetric with respect to that streamline. The quantities q^η and I_0, on the other hand, must be symmetric. With these considerations in mind we set

$$q^\xi \equiv [\xi/(1+\xi^2)]Q^\xi(z, \omega), \tag{26a}$$

$$q^\eta \equiv [1/(1+\xi^2)]Q^\eta(z, \omega) \tag{26b}$$

and

$$I_0 \equiv [1/(1+\xi^2)]J(z, \omega). \tag{26c}$$

The precise form of the functions in brackets on the right-hand side of these definitions is arbitrary, though a suitable choice will enhance the convergence of the expansions to be introduced later. As will be evident from the numerical results (this will be discussed further in §5), the functions used here serve reasonably well, although they may not necessarily be the best possible choice.

The governing equations in terms of the normalized variables are:

$$\begin{aligned}
&2z(1-z)\,\eta_{z\omega}-2\omega\eta_{\omega\omega}-2\eta_\omega+\frac{2\eta_\omega}{\eta}[z(1-z)\,\eta_z-\omega\eta_\omega]+\frac{2\eta_\omega}{P}[z(1-z)P_z-zP-\omega P_\omega]\\
&\quad-\frac{2\eta_\omega}{\Theta}[z(1-z)\Theta_z-z\Theta-\omega\Theta_\omega]+\eta_\omega+\frac{\eta_\omega z}{[z+\eta^2(1-z)]}+\frac{4\eta_\omega(1-z)\,[z(1-z)\eta_z-\omega\eta_\omega]^2}{[z+\eta^2(1-z)]}\\
&\quad-32\eta^2[z+\eta^2(1-z)]\,\frac{\eta_\omega^2 P\gamma M_r^2(1-z)}{(\gamma+1)^2\Theta}\{[(1-z)P_z-P]\eta_\omega-(1-z)\eta_z P_\omega\}=0, \qquad (27)
\end{aligned}$$

$$\begin{aligned}
&2\eta_\omega(1-z)[2z^2(1-z)\eta_{zz}+z(1-4z)\,\eta_z]-4z(1-z)[\omega\eta_\omega+z(1-z)\,\eta_z]\,\eta_{z\omega}\\
&\quad+4z(1-z)\,\eta_z\,\omega\eta_{\omega\omega}+2\eta_\omega[\omega\eta_\omega+2z(1-z)\,\eta_z]-4\eta_\omega\eta^{-1}[z(1-z)\,\eta_z-\omega\eta_\omega]^2\\
&\qquad\qquad+4\eta_\omega P^{-1}[\omega\eta_\omega-z(1-z)\,\eta_z]\,[z(1-z)P_z-zP-\omega P_\omega]\\
&\quad-4\eta_\omega\Theta^{-1}[\omega\eta_\omega-z(1-z)\,\eta_z]\,[z(1-z)\Theta_z-z\Theta-\omega\Theta_\omega]+2\eta_\omega[\omega\eta_\omega-z(1-z)\,\eta_z]\\
&\quad-\frac{\eta\eta_\omega z}{[z+\eta^2(1-z)]}-\frac{4\eta\eta_\omega(1-z)\,[\omega\eta_\omega-z(1-z)\,\eta_z]^2}{z+\eta^2(1-z)}\\
&\quad+\frac{16\eta^2[z+\eta^2(1-z)]\,\eta_\omega^2 P_\omega P\gamma M_r^2}{(\gamma+1)^2\Theta}=0, \qquad (28)
\end{aligned}$$

$$\begin{aligned}
&\gamma P[z(1-z)\Theta_z-z\Theta-\omega\Theta_\omega]-(\gamma-1)\Theta[z(1-z)P_z-zP-\omega P_\omega]\\
&\quad-\beta\Gamma\gamma M_r^2[2/(\gamma+1)]^a(\gamma-1)\,P^{a+1}\Theta^b(1-z)^{a+b-1}\,[z+\eta^2(1-z)]\\
&\qquad\times[J-4(1-z)^3\Theta^4]\,\eta\eta_\omega=0, \qquad (29)
\end{aligned}$$

$$\begin{aligned}
&\eta[2z(1-z)\,Q_z^\xi\,\eta_\omega-(2z-1)\,Q^\xi\eta_\omega-2z(1-z)\,\eta_z Q_\omega^\xi+Q_\omega^\eta]+[\eta Q^\xi+Q^\eta]\,\eta_\omega\\
&\qquad+\frac{\eta[zQ^\xi+\eta(1-z)\,Q^\eta]\,\eta_\omega}{[z+\eta^2(1-z)]}\\
&\quad+\beta\left(\frac{2P}{\gamma+1}\right)^a\Theta^b(1-z)^{a+b}\,[J-4(1-z)^3\Theta^4]\left[\frac{z+\eta^2(1-z)}{(1-z)}\right]^{\frac{1}{2}}\eta=0, \qquad (30)
\end{aligned}$$

$$[(1-z)\,J_z-J]\,\eta_\omega-(1-z)\,\eta_z J_\omega+\tfrac{3}{2}\beta\left(\frac{2P}{\gamma+1}\right)^a\Theta^b(1-z)^{a+b-1}\left[\frac{z+\eta^2(1-z)}{(1-z)}\right]^{\frac{1}{2}}\eta_\omega Q^\xi=0, \qquad (31)$$

and

$$J_\omega+3\beta\left(\frac{2P}{\gamma+1}\right)^a\Theta^b(1-z)^{z+b}\left[\frac{z+\eta^2(1-z)}{(1-z)}\right]^{\frac{1}{2}}\eta_\omega Q^\eta=0. \qquad (32)$$

In terms of the new variables the boundary conditions ($9c,d$), (17) and (10) immediately behind the shock ($\omega=1$) are

$$P(z,1)=\Theta(z,1)=\eta(z,1)=1,\quad \eta_\omega(z,1)=\tfrac{1}{2}[(\gamma-1)/(\gamma+1)], \qquad (33a,b)$$

$$J(z,1)-2Q^\eta(z,1)=0. \qquad (33c)$$

The condition (12) at the surface of the body ($\omega=0$) becomes

$$J(z,0)+\frac{2[Q^\eta(z,0)-2z(1-z)\,\eta_z(z,0)\,Q^\xi(z,0)]}{\{1+4z(1-z)^3[\eta_z(z,0)]^2\}^{\frac{1}{2}}}=0, \qquad (33d)$$

where we have now taken the wall temperature T_w to be zero, representing a highly cooled body.

We are now ready to apply the method of series truncation. Each dependent variable is first expanded in power series in z according to

$$F(z,\omega) = F_1(\omega)+zF_2(\omega)+\ldots, \tag{34}$$

where F represents any of the variables η, P, Θ, Q^ξ, Q^η or J. The subscripts 1 and 2 on the right-hand side of (34) correspond to the first- and second-order problems respectively. Substituting (34) into (27) to (32) and collecting like powers of z, we obtain for the first-order problem the following set of ordinary differential equations, where the dot denotes differentiation with respect to ω:

$$2\omega\ddot{\eta}_1-\frac{4\dot{\eta}_1^3\omega^2}{\eta_1^2}+\frac{2\omega\dot{\eta}_1^2}{\eta_1}+\dot{\eta}_1\left[1+\frac{2\omega\dot{P}_1}{P_1}-\frac{2\omega\dot{\Theta}_1}{\Theta_1}\right] \\ +\frac{32\eta_1^4P_1}{\Theta_1}\frac{\gamma M_r^2}{(\gamma+1)^2}\dot{\eta}_1^2[(P_2-P_1)\dot{\eta}_1-\eta_2\dot{P}_1] = 0, \tag{35}$$

$$\left[\frac{4\gamma M_r^2P_1\eta_1^4}{(\gamma+1)^2\Theta_1}-\frac{\omega^2}{P_1}\right]\dot{P}_1+\omega-\frac{2\omega^2\dot{\eta}_1}{\eta_1}+\frac{\omega^2\dot{\Theta}_1}{\Theta_1} = 0, \tag{36}$$

$$\gamma\omega P_1\dot{\Theta}_1-(\gamma-1)\omega\Theta_1\dot{P}_1+\beta\Gamma\gamma M_r^2[2/(\gamma+1)]^a(\gamma-1)P_1^{a+1}\Theta_1^b \\ \times[J_1-4\Theta_1^4]\eta_1^3\dot{\eta}_1 = 0, \tag{37}$$

$$\eta_1\dot{Q}_1^\eta+2(\eta_1Q_1^\xi+Q_1^\eta)\dot{\eta}_1+\beta\left(\frac{2P_1}{\gamma+1}\right)^a\Theta_1^b[J_1-4\Theta_1^4]\eta_1^2\dot{\eta}_1 = 0, \tag{38}$$

$$[J_2-J_1]\dot{\eta}_1-\eta_2\dot{J}_1+\tfrac{3}{2}\beta\left(\frac{2P_1}{\gamma+1}\right)^a\Theta_1^b\eta_1Q_1^\xi\dot{\eta}_1 = 0, \tag{39}$$

and

$$\dot{J}_1+3\beta\left(\frac{2P_1}{\gamma+1}\right)^a\Theta_1^bQ_1^\eta\eta_1\dot{\eta}_1 = 0. \tag{40}$$

The corresponding first-order boundary layer conditions obtained from (33) are

$$P_1(1) = \Theta_1(1) = \eta_1(1) = 1, \tag{41a}$$

$$\dot{\eta}_1(1) = \tfrac{1}{2}(\gamma-1)/(\gamma+1), \tag{41b}$$

$$J_1(1)-2Q_1^\eta(1) = 0, \tag{41c}$$

and

$$J_1(0)+2Q_1^\eta(0) = 0. \tag{41d}$$

It follows from (34) when written for $F\equiv\eta$ that in the first-order problem the body ($\omega = 0$) is a paraboloid (η = constant). The departure of the body from a paraboloid, as represented by the terms containing $\eta_z(z,0)$ in (33*d*), is therefore not reflected in the condition (41*d*).

The elliptic nature of both the subsonic flow field and the radiation field is shown by the appearance of the second-order quantities P_2 and η_2 in (35) and η_2 and J_2 in (39). Equations (35) to (40) as they stand thus do not constitute a determinate set, since the number of unknowns exceeds the number of equations. To obtain a determinate set, we arbitrarily take P_2, η_2 and J_2 equal to zero. This is equivalent to truncating each of the series (34) after the first term. The first-order truncation thus leads to a set of six ordinary differential equations with two-point boundary conditions. Similarly, the second-order truncation, which retains the first two terms in the series, leads to twelve ordinary differential equations, again with two-point boundary conditions. Since these equations are very lengthy, we shall not set them down explicitly.

4. Numerical integration

The ordinary differential equations obtained as above are to be integrated from the shock ($\omega = 1$) to the body ($\omega = 0$). This has been done numerically on an IBM 7090 computer by means of an Adams predictor-corrector routine with a Runge–Kutta starting procedure. The optimum step size for the specified error tolerance was chosen for each integration step by an error-control routine included in the program. The two-point boundary-value problem is converted to an initial-value problem by guessing the value of one of the unknown quantities at the shock. For example, for the first-order truncation, we guess the value of $Q_1^\eta(1)$. The value of $J_1(1)$ is then obtained from the boundary condition (41 c), and $Q_1^\xi(1)$ is found from the differential equation (39) with $\eta_2 = J_2 = 0$ and the resulting equation specialized to $\omega = 1$. All quantities immediately behind the shock are now known, and the numerical integration can proceed. Since our initial guess of the value of $Q_1^\eta(1)$ may not be accurate, the condition (41 d) at the body will not in general be satisfied. The value of $Q_1^\eta(1)$ must therefore be corrected. To carry out the correction systematically, Newton's method of iteration (see a standard text on numerical integration, e.g. Ostrowski 1960) was imbedded in the computer program. For the second-order truncation, we must guess the values of both $Q_1^\eta(1)$ and $Q_2^\eta(1)$. The procedures, however, are generally similar to those for the first-order truncation.

Four parameters appear in the differential equations and the boundary conditions, namely, γ, M_r, β and Γ. As a consequence of the strong-shock approximation, M_r and γ are related by the equation $M_r^2 = (\gamma+1)^2/2\gamma(\gamma-1)$. The specification of γ, β and Γ is thus sufficient to fix the problem. It will be convenient in presenting the numerical results to replace β by a Bouguer number Bu, defined in general as the ratio of a characteristic length in the flow field to a characteristic radiative mean free path. In particular we set $Bu \equiv \bar{\alpha}_s \bar{\Delta}$, where $\bar{\alpha}_s$ is the absorption coefficient (reciprocal of the radiative mean free path) immediately behind the shock on the stagnation streamline and $\bar{\Delta}$ is the stand-off distance. It is easy to show that Bu and β are related by the equation $Bu = [2/(\gamma+1)]\Delta\beta$, where the dimensionless stand-off distance $\Delta \equiv \bar{\Delta}/\bar{R}_s$ is obtained from the solution as a function of the parameters of the problem.

The numerical integration was carried out for a diatomic gas ($\gamma = \frac{7}{5}$) for three combinations of Γ and β, which give the following combinations of Γ and Bu:

$$
\begin{aligned}
&(1)\ \Gamma = 0{\cdot}4, && Bu = 2{\cdot}7,\\
&(2)\ \Gamma = 0{\cdot}4, && Bu = 5{\cdot}4,\\
&(3)\ \Gamma = 0{\cdot}04, && Bu = 0{\cdot}08.
\end{aligned}
$$

Results were also obtained for the non-radiating case (Γ or $Bu = 0$). To translate the foregoing figures into typical dimensional terms, we can consider air, for which $R = 2{\cdot}88 \times 10^2$ m²/sec² °K. Appropriate constants in the power-law representation for $\bar{\alpha} = C\bar{p}^a\bar{T}^b$, as used by Traugott (1963), are then $a = 1$, $b = 4$, and $C = 5{\cdot}37 \times 10^{-25}$ sec²/g °K⁴, when $\bar{p}$ is in dyne/m² and $\bar{T}$ is in °K. If the free-stream velocity is taken to be $\bar{U}_\infty = 5500$ m/sec, the temperature $\bar{T}_s$ given by the

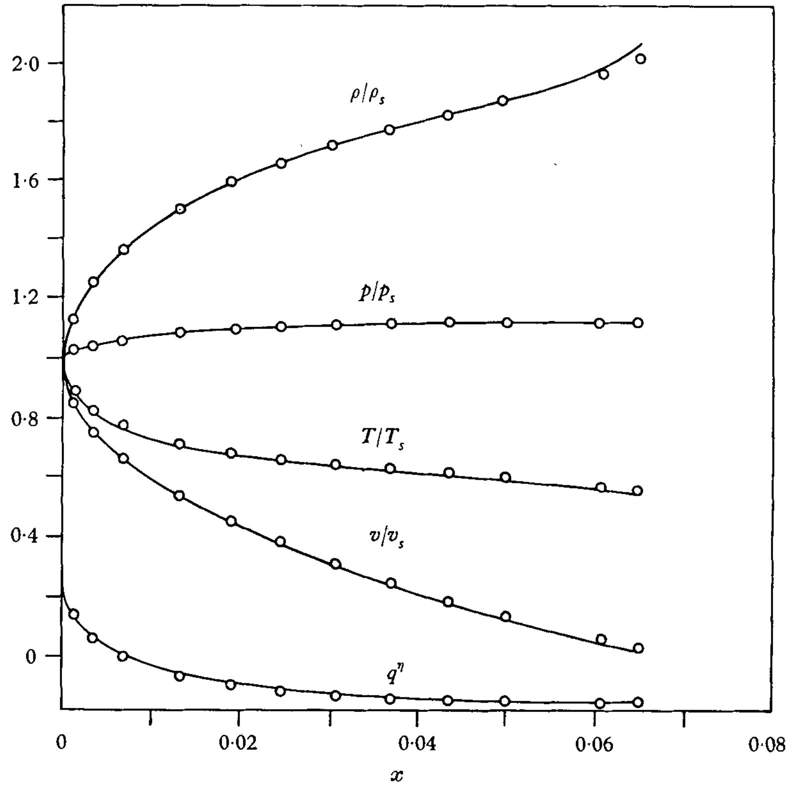

FIGURE 2. Comparison of flow and radiative variables along stagnation streamline as obtained from first- and second-order truncations for $\Gamma = 0{\cdot}4$, $Bu = 2{\cdot}7$, $\xi = 0$; ○, first-order truncation; ——, second-order truncation.

strong-shock approximation is $\bar{T}_s = 2(\gamma-1)\,U_\infty^2/(\gamma+1)^2 R \simeq 15{,}000$ °K. The values of $\bar{p}_s$ and $\bar{R}_s$ corresponding to the above cases then come out as follows:

(1) $\bar{p}_s = 1{\cdot}013 \times 10^9$ dyne/m² (10 atm), $\bar{R}_s = 1{\cdot}525$ m (5 ft.);

(2) $\bar{p}_s = 1{\cdot}013 \times 10^9$ dyne/m² (10 atm), $\bar{R}_s = 3{\cdot}050$ m (10 ft.);

(3) $\bar{p}_s = 1{\cdot}013 \times 10^{10}$ dyne/m² (100 atm), $\bar{R}_s = 0{\cdot}00305$ m (0·01 ft.).

We see from cases (1) and (2) that for a given gas and a given value of $\bar{U}_\infty$ (and hence fixed $\bar{T}_\infty$) an increase in Bu at fixed Γ (and hence fixed $\bar{p}_\infty$) corresponds to an increase in the scale of the flow field.

Computation times for the three cases ranged from 3 min to 3 h, depending on several factors. Long computation times are associated with steep gradients near the shock and the body, as well as with a slow convergence of the iteration process for certain combinations of the parameters. If temperature were used as the independent variable in the region where rapid changes in that quantity occur (see Conti 1966), the time would be considerably reduced. Computation time is also very much influenced by the initial guess of $Q_1^\eta(1)$ and $Q_2^\eta(1)$. If the guess is too far off, in fact, the iteration does not converge at all.

5. Results and discussion

Figure 1 shows the overall flow field for the non-radiating gas ($Bu = 0$) and for a typical example of the radiating gas ($\Gamma = 0{\cdot}4$, $Bu = 2{\cdot}7$). The effect of radiation on the dimensionless stand-off distance and on the shape of the body

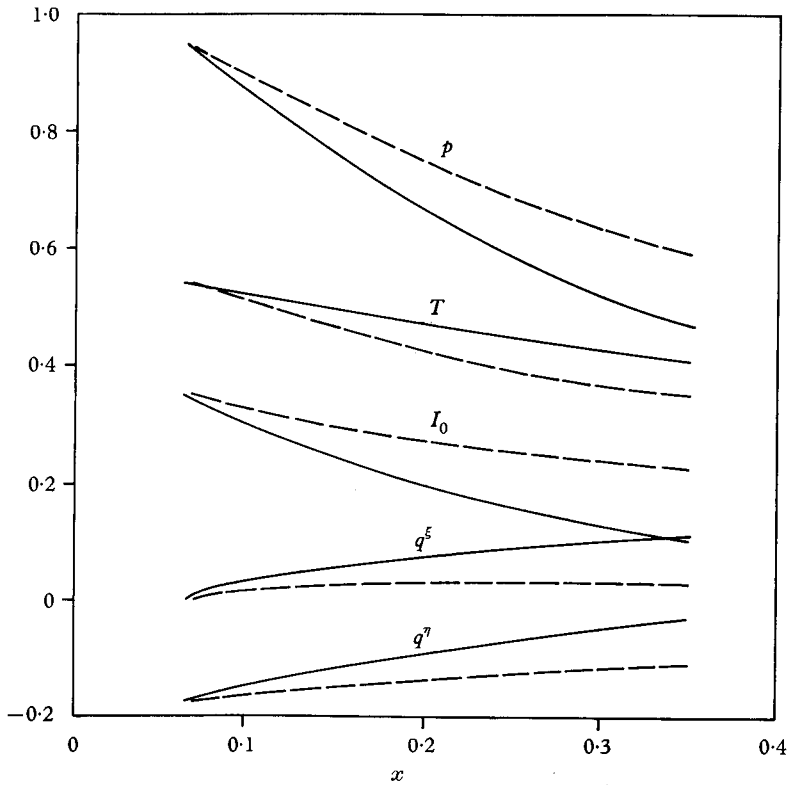

FIGURE 3. Comparison of flow and radiative variables along body surface as obtained from first- and second-order truncations for $\Gamma = 0{\cdot}4$, $Bu = 2{\cdot}7$; – – – –, first-order truncation; ——, second-order truncation.

and sonic line is apparent. The stand-off distance decreases considerably as the result of radiation, while the body shape is less affected.

Although no analytical method can be provided for estimating the error in the truncation scheme, some measure of its accuracy can be obtained by comparing the results of the first- and second-order truncations. This is done in figure 2 for quantities on the stagnation streamline for $\Gamma = 0{\cdot}4$, $Bu = 2{\cdot}7$. Here the improvement given by the second-order truncation is hardly discernible, except near the stagnation point, where the two truncations disagree in the third significant figure. A similar comparison of quantities on the surface of the body is given in figure 3. As would be expected, the disagreement grows with distance from the stagnation point. At $x = 0{\cdot}35$, for example, the second-order results reduce the calculated pressure by 25 %. This is about the same as in the non-radiating-gas

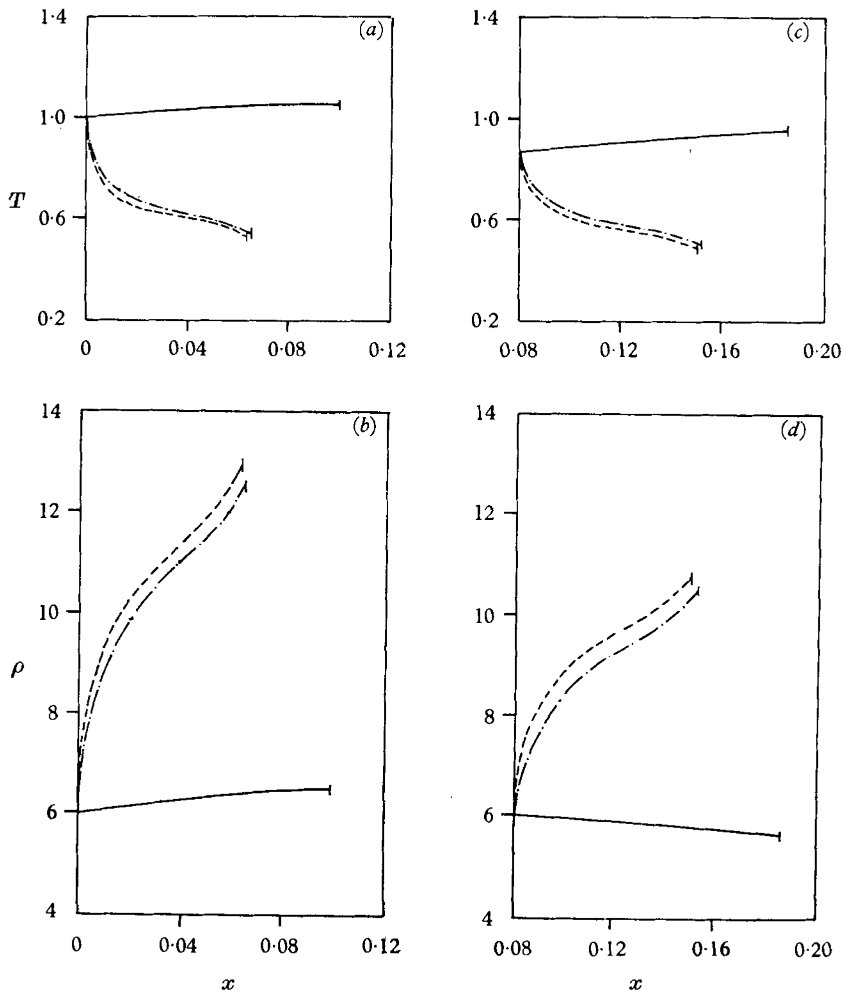

FIGURE 4. Effect of radiation on temperature and density distribution across shock layer for $\Gamma = 0{\cdot}4$ and various values of *Bu*. (For explanation of $\Gamma \equiv \sigma \overline{T}_s^4/\rho_\infty \overline{U}_\infty^3$, see §2, after (5); and of $Bu \equiv \overline{\alpha}_s \overline{\Delta}$, see §4.) (*a*) $\xi = 0$; (*b*) $\xi = 0$; (*c*) $\xi = 0{\cdot}4$; (*d*) $\xi = 0{\cdot}4$; $\Gamma = 0{\cdot}4$, ——, $Bu = 0$; —·—, $Bu = 2{\cdot}7$; ----, $Bu = 5{\cdot}4$.

results of Van Dyke (1965), where going on to the third- and fourth-order truncations was found to give negligible further improvement. For the radiative quantities, the first-order results at the same location overestimate the magnitude of I_0 by 100 % and q^η by 200 % and underestimate q^ξ by 75 %. This suggests that the second-order results for these quantities may be somewhat inaccurate near the sonic line. To improve this situation one could either carry out higher-order truncations or improve the convergence of the second-order truncation by changing the form of the bracketed functions in (26). The disagreement noted in figure 3 suggests that a better choice might be $q^{\xi} \equiv \xi Q^{\xi}(z, \omega)$, $q^{\eta} \equiv [1/(1+\xi^2)^2] Q^{\eta}(z, \omega)$ and $I_0 \equiv [1/(1+\xi^2)^2] J(z, \omega)$. Neither of these methods has been attempted here.

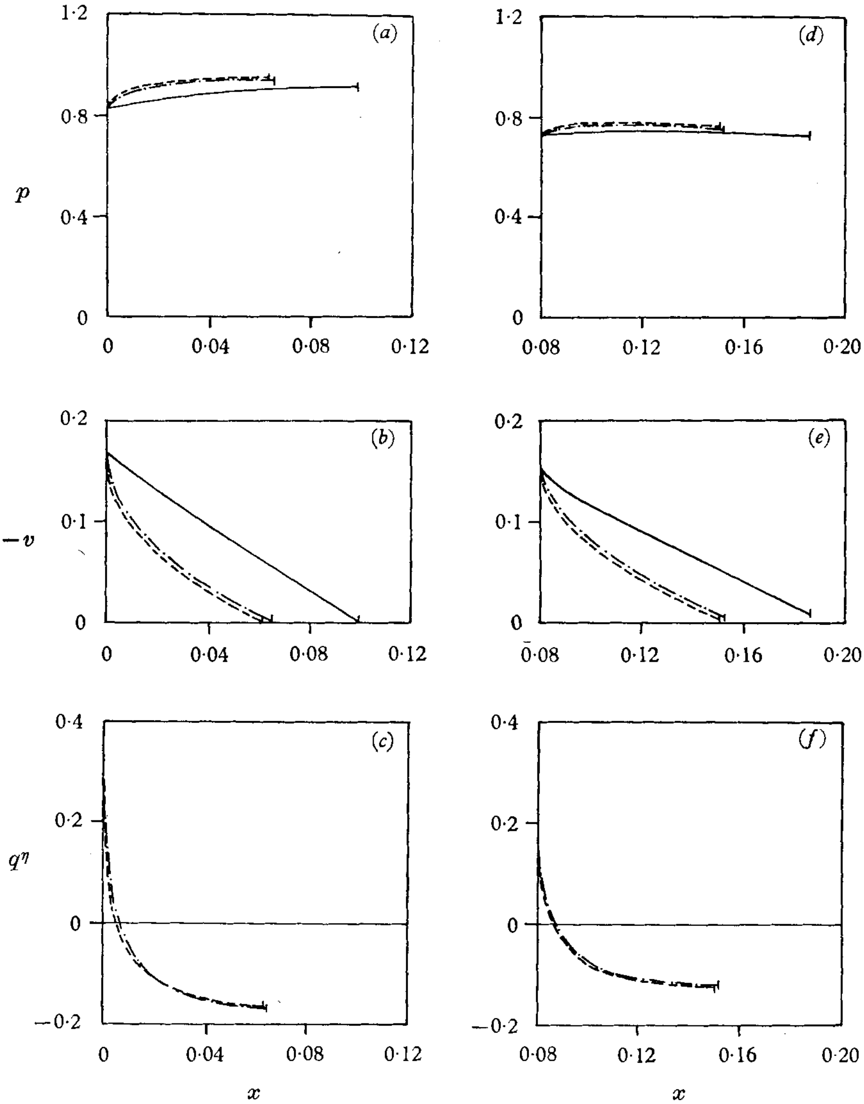

FIGURE 5. Effect of radiation on pressure, normal velocity component, and normal heat-flux component across shock layer for $\Gamma = 0{\cdot}4$; ——, $Bu = 0$; —·—, $Bu = 2{\cdot}7$; ----, $Bu = 5{\cdot}4$. (*a*) $\xi = 0$; (*b*) $\xi = 0$; (*c*) $\xi = 0$; (*d*) $\xi = 0{\cdot}4$; (*e*) $\xi = 0{\cdot}4$; (*f*) $\xi = 0{\cdot}4$.

The effect of radiation on the flow and radiative quantities across the shock layer for $\Gamma = 0{\cdot}4$ and three values of Bu is shown in figures 4–6. Radiation is seen to cause a rapid drop in temperature and a rapid rise in density immediately behind the shock (figure 4). Since viscosity and thermal conductivity have been neglected, the temperature of the gas adjacent to the wall need not be equal to that of the wall (taken to be zero in this work). The resulting temperature jump is apparent in figures 4*a* and *c*.

For a given value of x, the effect of radiation is to cause a small increase in pressure (figures 5*a* and *d*) and a decrease in the normal velocity v (figures 5*b* and *e*). (For convenience we refer to v and u as the normal and tangential velocity

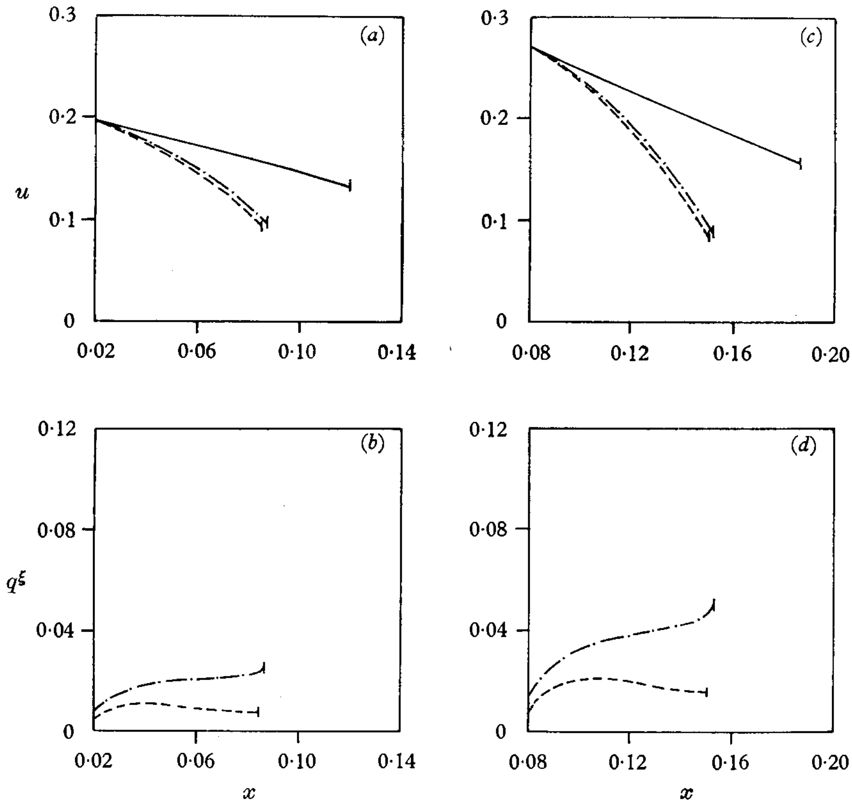

FIGURE 6. Effect of radiation on tangential velocity component and tangential heat-flux component across shock layer for $\Gamma = 0{\cdot}4$; ——, $Bu = 0$; —·—, $Bu = 2{\cdot}7$; ---, $Bu = 5{\cdot}4$. (*a*) $\xi = 0{\cdot}2$; (*b*) $\xi = 0{\cdot}2$; (*c*) $\xi = 0{\cdot}4$; (*d*) $\xi = 0{\cdot}4$.

and to q^η and q^ξ as the normal and tangential heat flux.) The decrease in normal velocity is a direct consequence of the decrease in stand-off distance. The normal heat flux (figures 5*c* and *f*) is positive (i.e. away from the body) immediately behind the shock, decreases rapidly to zero, and then reverses and increases in magnitude as the surface of the body is approached. This behaviour is consistent with the temperature distribution of figures 4*a* and *c*. At the shock the heat flux from the hot gas in the shock layer must obviously be outward toward the free stream; at the body it must be inward toward the cold body. The reversal that must therefore appear is seen here to occur relatively close to the shock. On the stagnation streamline ($\xi = 0$) the magnitude of the normal heat flux is larger at the shock than at the wall (for zero wall temperature). This agrees with the results of Yoshikawa & Chapman (1962) and Wang (1964).

As shown in figures 6*a* and *c*, radiation causes a decrease in the tangential velocity at a given location. Even in the presence of radiation, the variation in tangential velocity remains nearly linear with respect to position in the shock

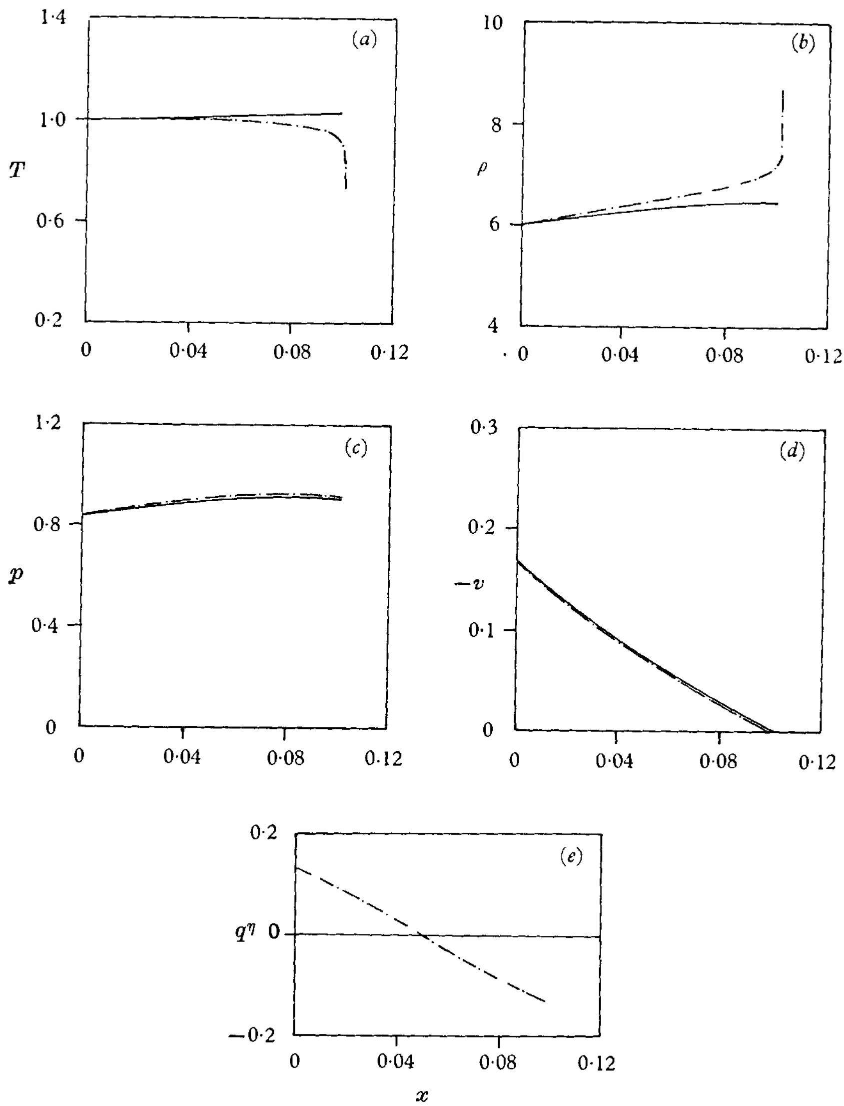

FIGURE 7. Effect of radiation on temperature, density, pressure, normal velocity component, and normal heat-flux component along stagnation streamline for $\Gamma = 0{\cdot}04$, $Bu = 0{\cdot}08$; ——, $Bu = 0$; —·—, $Bu = 0{\cdot}08$. (*a*) $\xi = 0$; (*b*) $\xi = 0$; (*c*) $\xi = 0$; (*d*) $\xi = 0$; (*e*) $\xi = 0$.

layer. This supports the assumption of a linear tangential-velocity profile in the Dorodnitsyn variable in the work of Wilson & Hoshizaki (1965). Inside the shock layer the tangential heat flux is always positive (figures 6*b* and *d*). This is consistent with the results of figures 4*a* and *c*, which show that in general the temperature level decreases as ξ increases. Comparing figures 5*c* and *f* with figures 6*b* and d, we see that the normal heat flux is much larger than the tangential heat flux everywhere except in a small region near the point at which the normal heat flux vanishes.

Figure 7 shows the distribution of various quantities along the stagnation streamline for especially small values of Γ and Bu (0·04 and 0·08 respectively). This corresponds to an optically thin situation with the radiative flux of energy

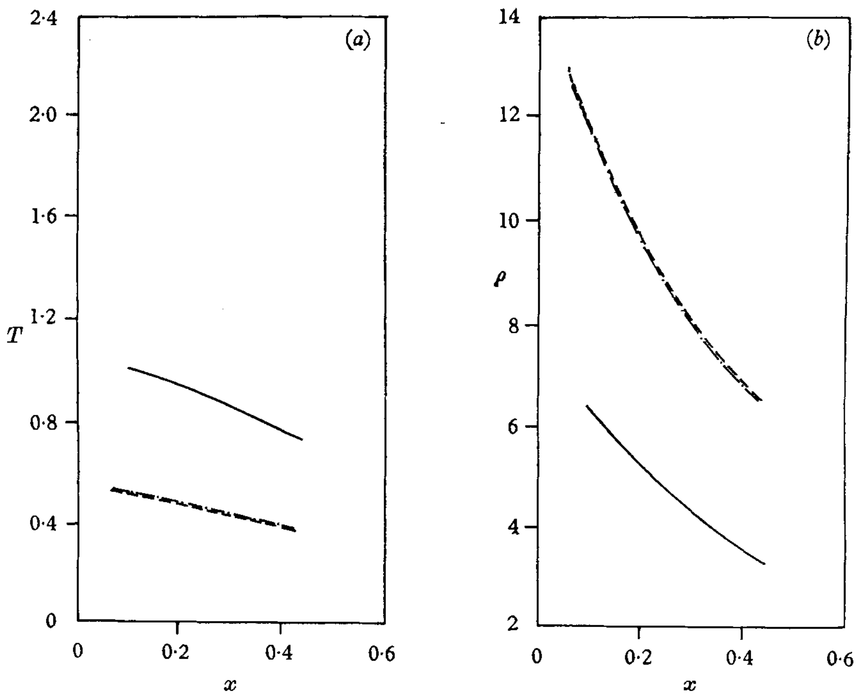

FIGURE 8. Effect of radiation on gas temperature and density along body surface for $\Gamma = 0{\cdot}4$; ——, $Bu = 0$; —·—, $Bu = 2{\cdot}7$; ---, $Bu = 5{\cdot}4$.

very small relative to the convective flux. With regard to temperature and density (figures 7*a* and *b*), the rapid changes that exist immediately behind the shock in the earlier cases (cf. figures 4*a* and *b*) do not appear here. The effect of radiation on these quantities is in fact small except near the body, where there is a sharp fall in temperature and rise in density. This behaviour has been studied by a perturbation in terms of essentially the product $Bu\Gamma$ by Olstad (1965), who uses the Lighthill technique to obtain a uniformly valid solution. As seen in figure 7*a* the temperature of the gas next to the wall is again different from the zero temperature of the wall itself. The assumption of no reabsorption, which is made in the thin-gas approximation and which leads to the erroneous result of zero gas temperature at the wall (Wilson & Hoshizaki 1965; Wang 1965), is not present here. The differential approximation as used here thus takes proper account of the reabsorption effects, which are of crucial importance in the strongly cooled region near the wall.

With regard to pressure and velocity (figures 7*c* and *d*), radiation is seen here to have practically no effect at all points on the stagnation streamline. The normal velocity exhibits a nearly linear variation with distance. In figure 7*e* the zero in the normal heat flux now occurs near the middle of the shock layer (cf. figure 5*c*). This is consistent with the fact that the temperature is nearly constant

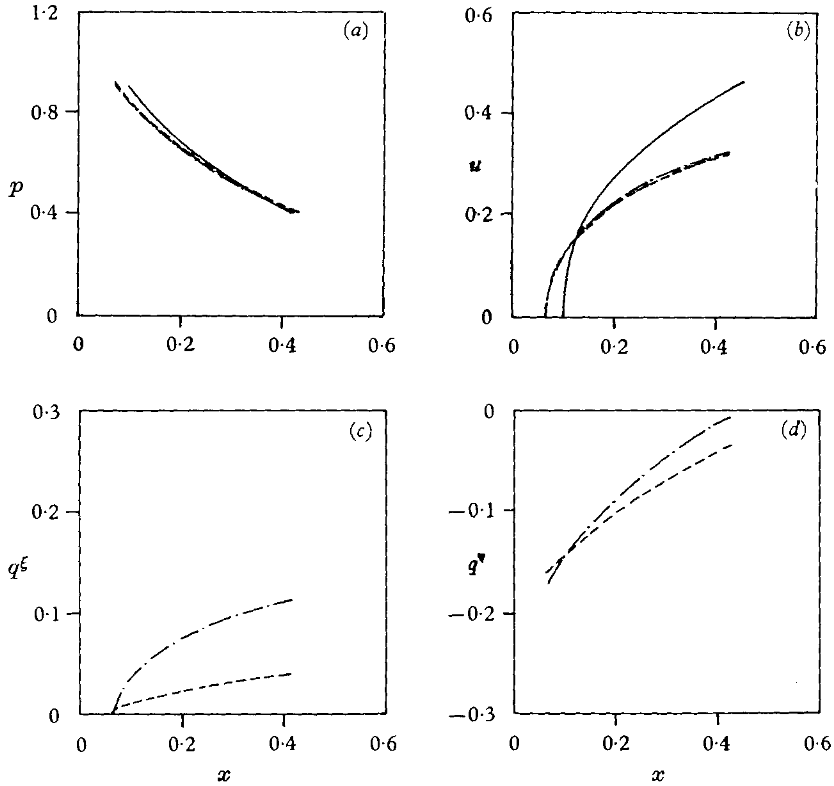

FIGURE 9. Effect of radiation on pressure, tangential velocity component, and heat-flux components along body surface for $\Gamma = 0{\cdot}4$. ——, $Bu = 0$; —·—, $Bu = 2{\cdot}7$; – – –, $Bu = 5{\cdot}4$.

across most of the layer, so that the radiative conditions have a kind of 'symmetry' with respect to the mid-point. The effect of optical thickness on the location of the zero in the heat flux has also been observed by Wang (1964), whose conclusions are in qualitative agreement with ours.

Figures 8 and 9 show the effect of radiation on the flow and radiative variables along the body surface for the earlier conditions of figures 4–6. In examining these results it should be remembered that in the present inverse problem the shape of the body changes slightly as the parameters are varied. Everywhere on the surface, the decrease in gas temperature and increase in density as the result of radiation are seen to be considerable (figure 8). The tangential heat flux (figure 9*c*) is everywhere positive, i.e. in the direction away from the stagnation point. The normal heat flux (figure 9*d*) is everywhere negative, i.e. into the wall, as would be expected with the wall at zero temperature. As the radiating gas flows along the surface, the tangential heat flux increases in magnitude and the normal heat flux decreases.

It is interesting finally to compare the present results with those of Lick (1960) and Conti (1966) for non-equilibrium reactive flows. Such comparison shows a marked similarity between the effects of radiation and of non-equilibrium chemical reactions. The shock-nose radius plays much the same role in the radiating flow at a fixed value of Γ as it does in the non-equilibrium reactive flow (cf. Conti 1966). In particular, the radiating flow with a large value of *Bu* has much in common with near-equilibrium flow; a small value of *Bu* corresponds to near-frozen flow. Thus, for example, the rapid changes in temperature near the stagnation point in radiating flow at a small value of *Bu* are similar to those observed in near-frozen flow. Both radiation and non-equilibrium chemical reactions cause an increase in density and pressure in the shock layer and a decrease in temperature, stand-off distance, and velocity. In both situations, density, temperature, and stand-off distance are relatively sensitive, whereas velocity is moderately affected and pressure hardly at all.

6. Concluding remarks

Our primary aim has been an understanding of the multidimensional effects in non-linear radiating flow, with no special assumptions regarding the optical thickness. We have made no attempt at a quantitative comparison between our results and those obtained in earlier studies on the basis of the thin-gas approximation or a one-dimensional treatment of the radiative field. Such comparison is hardly possible, since the earlier works deal with the direct rather than the inverse problem and use differing assumptions for the absorption coefficient and the state relations (most of them taking into account imperfect-gas effects). Our results, however, do agree qualitatively with those of other investigators in so far as they are mutually comparable.

The present work shows clearly the utility of the differential approximation in radiative gas dynamics. With this approximation, the multidimensional and re-absorption effects can simultaneously be taken into account. The results have a physically realistic behaviour throughout the flow field for the entire range of optical thickness.

The usefulness of the method of series truncation as a mathematical tool for studying the blunt-body problem is also illustrated. By means of this method, the details of the flow and radiation fields can be obtained with reasonable ease and no further mathematical assumptions. The numerical results thus provide a check on the assumptions made in other more approximate treatments.

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